Interactions and Reconnections of Extra-Dimensional Quantum Vortices

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Interactions and reconnections of vortices are fundamental in many areas of physics, including in both classical and quantum fluids. The reconnection procedure provides the mechanism for the distribution of energy to multiple lengthscales throughout the system. This has been studied and observed in three-dimensional quantum fluids, resulting in the reporting of universal scaling laws [1]. We generalise this to a four-dimensional system to study how two vortices in a four-dimensional quantum fluids interact.

Recent rapid experimental progress in the creation of synthetic dimensions in ultra-cold atoms and molecules [2] and the recent achievement of a molecular Bose-Einstein condensate [3] provides ample reasoning to explore these extra-dimensional dynamics of topological defects in condensates, with a view of testing and exploring the current laws of universality in vortex reconnections [4].

Background and methods

Unlike the arbitrary circulation and size allowed in classical fluids, quantised vortices have fixed allowed values, with circulation taking quantised values, scaled by $q=h/m$, where m is the mass of the atomic species of the system. The width of the vortex core is scaled by the healing length of the system ξ. In three-dimensional systems, these vortices take the form of lines, whereas in four-dimensional systems, they are planes [5]. The reconnection process of two three-dimensional quantum vortices can be quantified by the minimum distance, δ, of the two vortices in time, t,

$$\delta(t) = \alpha^{\pm}|t - t_0|^{1/2}$$

where $\alpha^{\pm}$ denotes a demensionless scaling parameter with - (+) being before (after) the time of reconnection t_0. It has been shown that in three dimensional systems, that for all cases of reconnection $\alpha^+ > \alpha^-$ signifying that vortices repel faster than they approach.

In this work, we model our system as a bosonic quantum fluid in the low temperature limit. In this mean-field regime, the condensate can be modelled as a single wave function, ψ. We model the dynamics of vortices in a four-dimensional condensate. The Gross-Pitaevskii equation (GPE) in these four dimensions is,

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t} = \Big[-\frac{1}{2}\nabla^2 + V(\mathbf{x}) + |\psi(\mathbf{x},t)|^2\Big]\psi(\mathbf{x},t),$$

where $\mathbf{x} = (x,y,z,w)$ and V denotes the external trapping potential, we defined as a hard wall hyperspherical potential,

$$V(x,y,z,w) = \begin{cases} 0, & \text{if } r^2 < 30 \\ 10, & \text{if } r^2 \geq 30 \end{cases}$$

where $r = \sqrt{x^2 = y^2 + z^2 + w^2}$. The GPE is solved using the fourth-order Runge-Kutta scheme, with standard finite-difference methods. We work with dual vortex states, where two vortices are oriented with angles α and β. The vortices are placed in the system where the first vortex is aligned in the (x',y') plane and the second vortex is in the (z',w') plane such that

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & \cos\beta & 0 & -\sin\beta \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix},$$

For different values of α and β, the vortices can exhibit different rotational allignments, which we will show gives very different dynamics. Firstly is *aligning* case, where $\alpha = \beta$. Secondly, we show the *simple* rotating case where $\beta = 0$, and $|\alpha|>0$; this case is the analogous to the three-dimensional reconnection case. Finally, we present the anti-aligning case, where $\alpha = -\beta$. We express angles in this work as the difference between the angles of the two vortices $\theta = \beta - \alpha$. Visualisation of 4D vortices can have its difficulties; in this work we present the vortex structures physically in the (x, y, z) subspace, with the w direction denoted by the colour of the image.

Aligning vortices

When vortices exist in an aligning orientation, the vortices rotate around each other freely, without any inter-vortex interactions, akin to the Abrikosov lattice in two- and three-dimensional systems. This agrees with previously published work, solving the GPE equation in four dimensions in imaginary time with a rotational term added [5]. It can be shown that the two vortices rotate with a frequency, Ω, given by,

$$\Omega = \frac{4}{r^2}\frac{\tan\alpha}{\cos\beta}$$

We note here that two vortices in four dimensions have to touch either at a point or a plane, unlike in lower dimensional systems. In the alligning case, there is no reconnection, even when the vortices touch.

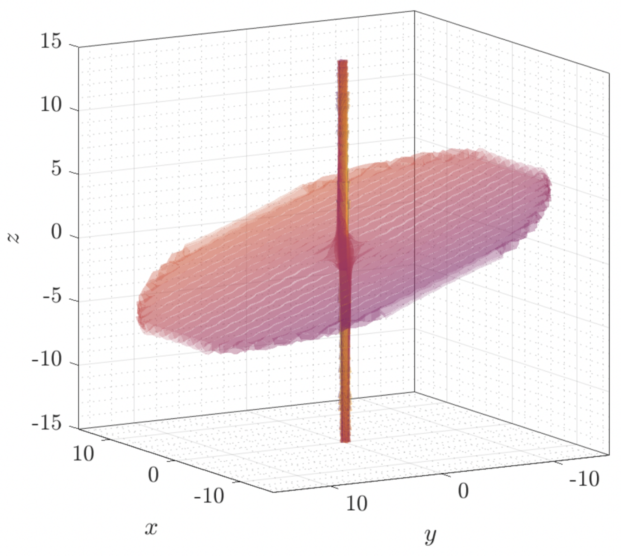

Fig 1: Two vorttices placed in an aligning orientation with θ=20°.

Simple rotations

The simple rotation case results in a vortex layout similar to the case one often sees in studies of reconnection in three-dimensional systems. That is, for $\theta=90°$, the two vortices will be alligned perpendicularly in the (x,y,z) subspace. An example of the vortex interaction for a case of $\theta=45°$ is shown in Fig. 2 at three different times.

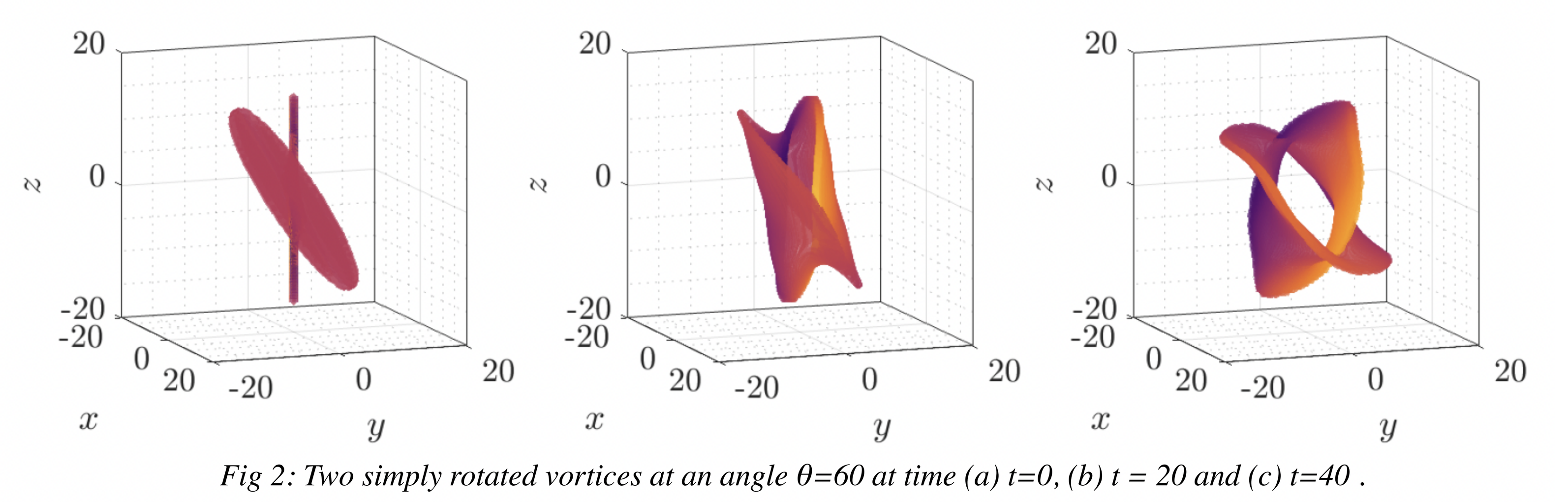

Fig 2: Two simply rotated vortices at an angle θ=60 at time (a) t=0, (b) t = 20 and (c) t=40 .

It is shown here that the vortices will reconnect akin to its three-dimensional counterpart. Unlike its counterpart, however, the vortices will continue to touch and will reconnect in a central region. It can be proven by a deformation retract that the reconnected region will be homotopic to a circle [4], which agrees with the results in Fig. 2, obtained numerically.

By measuring the minimum distance within this reconnected region, we can show that the same universal scaling law in Eq. (1) holds for reconnection in four dimensions.

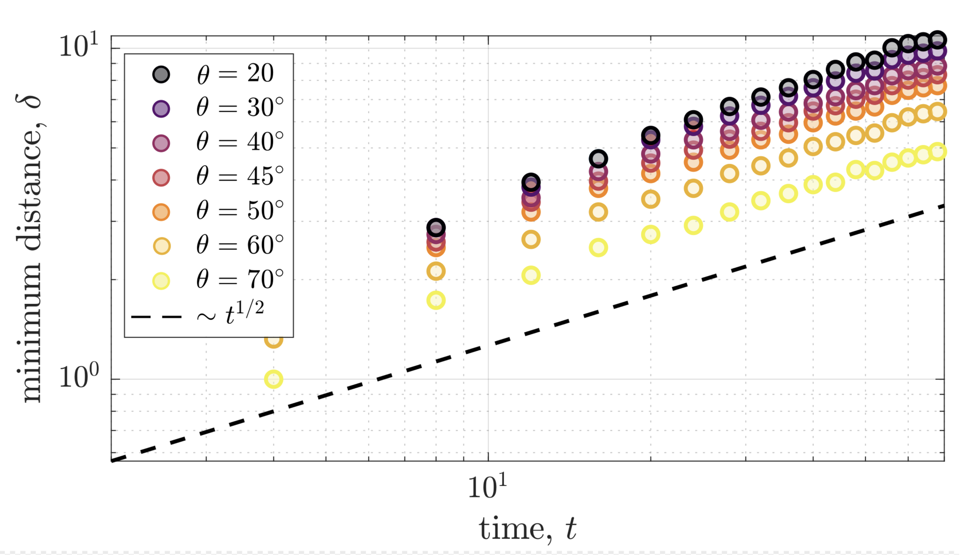

Fig 3: The minimum distance of the reconnection region as a function of time for orientation angles, θ with a t^1/2 scaling for reference.

As well as measuring the minimum distance, we can observe the change in the incompressible energy of the system. The incompressible energy is the kinetic energy corresponding to the quantum vortex. In three dimensions, this is converted to compressible energy - sound energy - during the reconnection process. The incomppressible energy is often obtained via the Helmholtz decomposition, which is not valid in four dimensions. Instead, the more generalised Hodge decomposition is used [4,5], which we use to show the incompressible energy fall after reconnection, showing the time irreversible dynamics reported in three dimensions (Fig. 4).

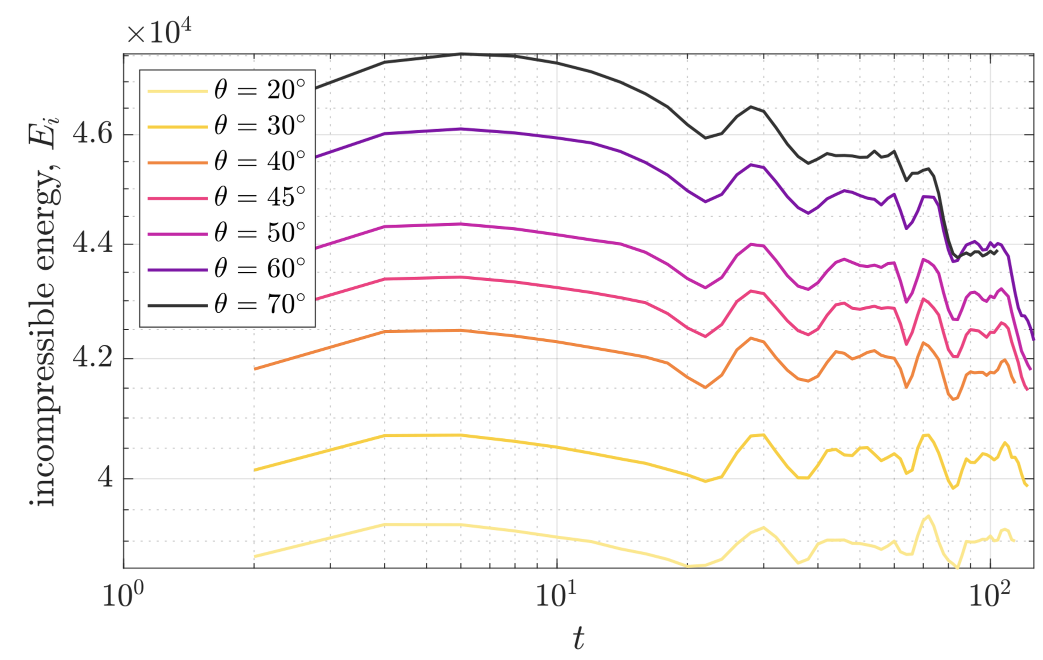

Fig 4: The loss of incompressible energy of the vortex state over time for different orientation angles.

Anti-aligning vortices

The result for two anti-alligning vortices is, on first glance, very similar to the previous simple rotation case. Observing Figs. (5) and (6), we notice again the circular reconnection zone, which expands out following the relation given in Eq. 1. One difference one can notice is the smoothness of Fig. 5c compared to it's counterpart in Fig 2c. - there is a lack of dynamical instability one typically associates with the reconnection process and the conversion of incompressible to compressible energy and the Kelvin waves formed.

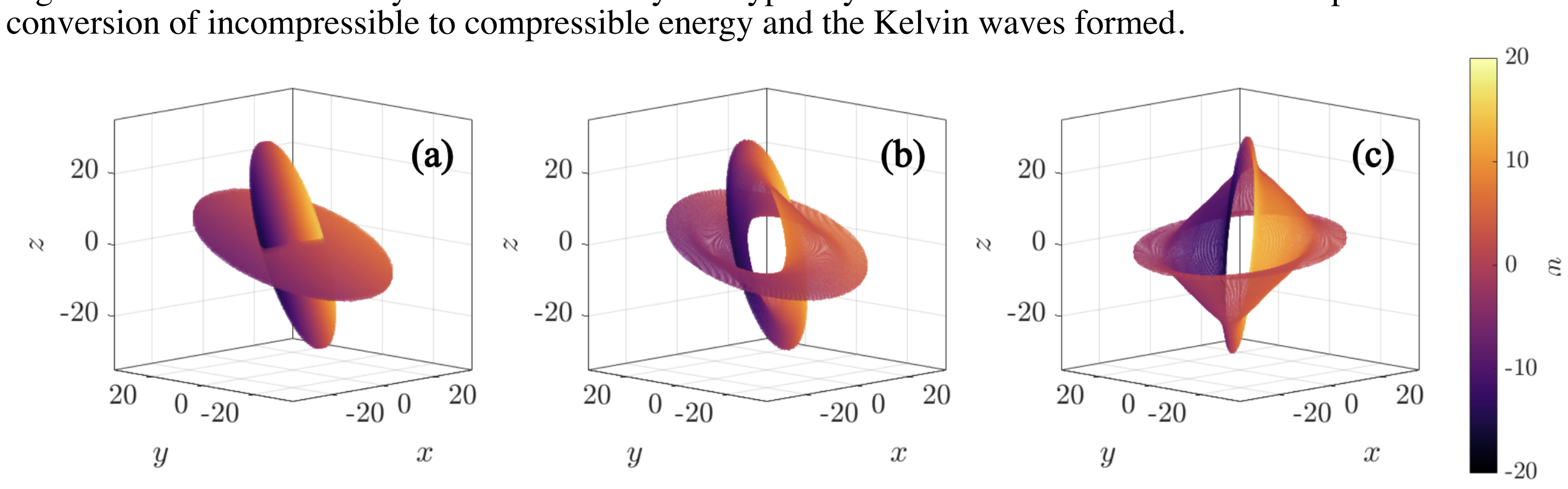

Fig 2: Two anti-aligning vortices at an angle θ=60 at time (a) t=0, (b) t = 20 and (c) t=40 .

Analysis of two vortices reconnecting multiple times over a long simulation shows a lack of instabilities forming, even after many passes. We also notice the quicker reconnection compared to the simple case in Fig. 3.

Calculating the incompressible energy of two anti-aligning vortices reconnecting multiple times we see in Fig. 7, that the incompressible energy does not fall. Instead, the energy simply fluctates by less than 1% around a mean value. This implies that the time reversable reconnections are possible for certain types of vortices in a four dimensional system.

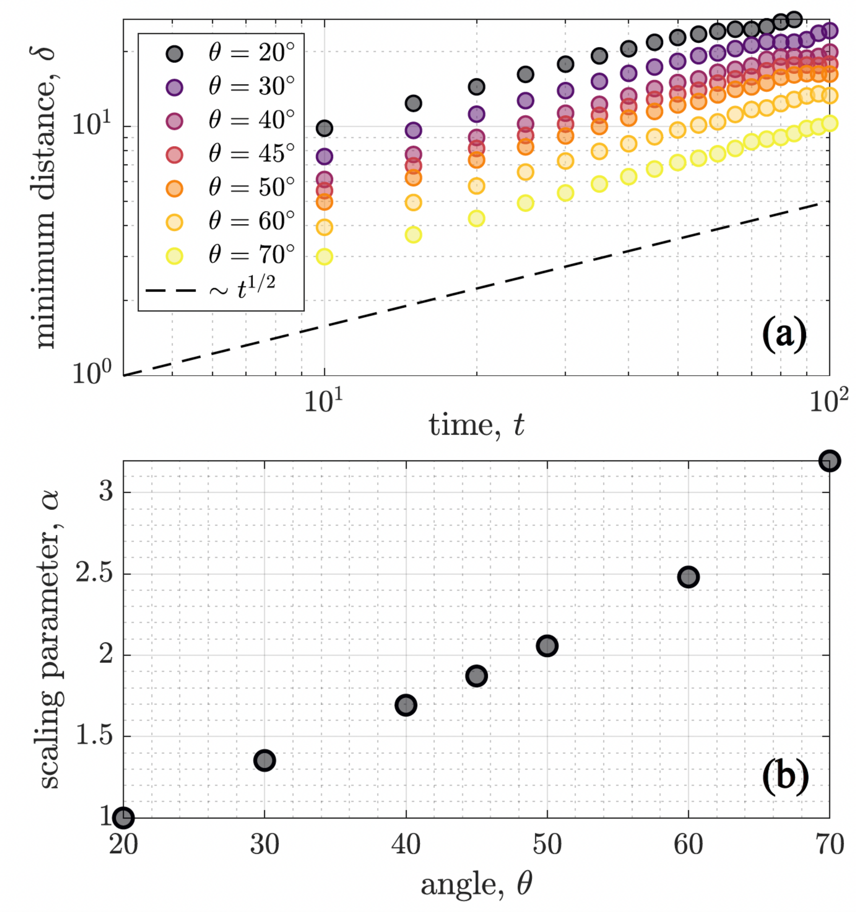

Fig 6: The minimum distance of the reconnection region as a function of time for orientation angles, θ with a t^1/2 scaling for reference, for two anti-alinging vortices and (b) the measured scaling parameter from Eq. 1.

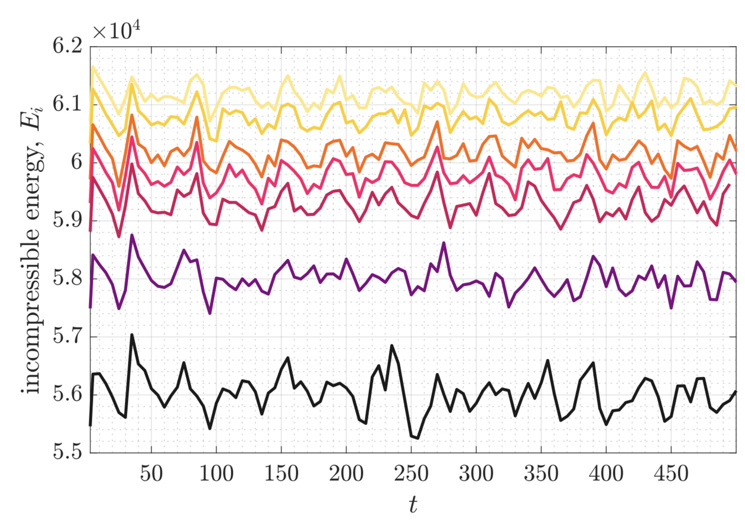

Fig 7: The incompressible energy for each orientation angle as a function of time; it fluctuates by around 1%, but never falls for a long simulation .

Conclusions and further work

In conclusion, we have introduced the three different ways that vortices can interact in a four dimensional system. We have shown that these have very different dynamics, with the anti-aligning case having no counterpart in the two- or three-dimensional systems. Unlike in the three-dimensional and simple reconnection, the lack of energy conversion implies that the reconnection mechanism is time reversible.

Further work consists of further investigating this anti-aligning case. We also will further analyse the different topologies vortices can take in four dimensional systems. In three dimensions, vortices can take the shape of lines or rings; we have focussed on vortex planes in this work, however other toplogies exist in four dimensions.

In cold atom systems, synthetic dimensions have . Further work is needed to find an experimentally viable way of sustaining four-dimensional vortices in a condensate with a synthetic dimension; coupling between states in the synthetic dimension can be complex and is discrete, unlike our model here.

References

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